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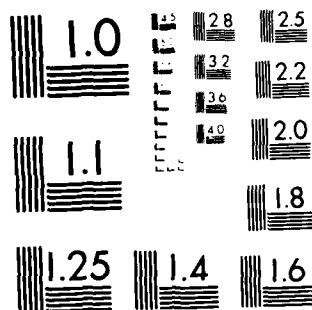
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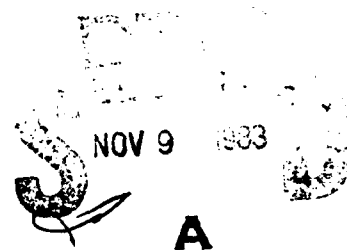
A VARIATIONAL APPROACH TO SUPERLINEAR
ELLIPTIC PROBLEMS

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ABSTRACT

This paper contains a variational treatment of the Ambrosetti-Prodi problem, including the superlinear case. The main result extends previous ones by Kazdan-Warner, Amann-Hess, Dancer, K. C. Chang and de Figueiredo. The required abstract results on critical point theory of functionals in Hilbert space are all proved using Ekeland's variational principle. These results apply as well to other superlinear elliptic problems provided an ordered pair of a sub- and a supersolution is exhibited.

AMS (MOS) Subject Classifications: 35J65, 47H15, 58E05

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SIGNIFICANCE AND EXPLANATION

Semilinear elliptic boundary value problems of the type

$$(*) \quad -\Delta u = g(u) + f(x) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega$$

where Ω is some bounded domain in \mathbb{R}^N and $g : \mathbb{R} \rightarrow \mathbb{R}$ and $f : \Omega \rightarrow \mathbb{R}$ are some given functions, have no solution in general if

$$(**) \quad \limsup_{u \rightarrow -\infty} \frac{g(u)}{u} < \lambda_1 < \liminf_{u \rightarrow +\infty} \frac{g(u)}{u},$$

where λ_1 is the first eigenvalue of $-\Delta$ in H_0^1 . However in 1972, Ambrosetti and Prodi observed a very interesting phenomenon concerning this class of problems. Namely, they showed that there is a manifold M in $C^{0,\alpha}(\Omega)$ which disconnects $C^{0,\alpha}(\Omega)$ into two open sets O_0 and O_2 such that if $f \in O_0$, problem (*) has no solution, if $f \in M$ problem (*) has exactly one solution, if $f \in O_2$ problem (*) has exactly two solutions. Their result was obtained under very stringent conditions on g . Namely, convexity, asymptotically linear growth at $\pm\infty$, and the second limit in (**) being less than the second eigenvalue of $-\Delta$ in H_0^1 . In subsequent years there has been an increasing effort to understand the actual role of these assumptions in this phenomenon. The present paper is a contribution in this direction. It is seen that as far as existence (or nonexistence) is concerned these assumptions are not essential. What matters is the crossing of the first eigenvalue λ_1 as stated in (**). The previous results in this subject by Amann and Hess and by Dancer use topological degree to obtain the second solution. For that purpose de Figueiredo used a variational argument obtaining a more general result. The first solution is always obtained in the preceding works by the method of monotone iteration, as does Kazdan and Warner. In the present paper we obtain much more complete results proceeding solely by variational methods.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

A VARIATIONAL APPROACH TO SUPERLINEAR ELLIPTIC PROBLEMS

Djairo G. de Figueiredo* and Sergio Solimini**

INTRODUCTION.

The main objective of the present paper is to give a variational treatment of the superlinear Ambrosetti-Prodi problem and to prove the existence of two solutions for certain values of the parameter. In this process we produce interesting and simple proofs of some abstract results using Ekeland's variational principle [1]. Some of these abstract results, namely Propositions 3 and 5, are known in the literature, where they are proved via the deformation lemma, c.f. Hofer [2] and Rabinowitz [3]. Our results on the superlinear Ambrosetti-Prodi problem extend previous theorems of Dancer [4], K. C. Chang [5] and one of the authors [6].

Let Ω be a smooth bounded domain in R^N , $N \geq 2$. Let $g : \Omega \times R \rightarrow R$ be a function satisfying the following conditions:

(g1) g is a C^1 -function,

(g2) $\limsup_{s \rightarrow -\infty} \frac{g(x,s)}{s} < \lambda_1 < \liminf_{s \rightarrow +\infty} \frac{g(x,s)}{s}$

(g3) $\lim_{s \rightarrow +\infty} \frac{g(x,s)}{s^\sigma} = 0$, where $1 < \sigma < (N+2)/(N-2)$ if $N \geq 3$
or $1 < \sigma < \infty$ if $N = 2$

(g4) $\liminf_{s \rightarrow +\infty} \frac{sg(x,s) - \theta G(x,s)}{s^2 g(x,s)^{2/(N+1)}} > 0$, if $N \geq 3$ and $(N+1)/(N-1) < \sigma < (N+2)/(N-2)$

where $\theta > 2$ and $G(x,s) = \int_0^s g(x,\xi) d\xi$.

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The statements about the limits above are supposed to hold uniformly for $x \in \Omega$. The limits in (g2) could assume value $-\infty$ or $+\infty$, respectively, on the whole of Ω or on subsets of positive measure. In (g2), λ_1 denotes the first eigenvalue of the eigenvalue problem $-\Delta u = \lambda u$ in Ω , $u = 0$ on $\partial\Omega$. We denote by ϕ the eigenfunction corresponding to λ_1 and such that $\phi > 0$ in Ω and $\int \phi^2 = 1$. Let $\phi^\perp = \{u \in C^0(\bar{\Omega}) : \int u\phi = 0\}$. Any function $f \in C^0(\bar{\Omega})$ can be uniquely decomposed as $f = t\phi + h$, where $t \in \mathbb{R}$ and $h \in \phi^\perp$. We shall look at the parametrized family of Dirichlet problems:

$$(P_t) \quad -\Delta u = g(x, u) + t\phi + h \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

and prove the following result.

Theorem 1. Assume (g1), (g2), (g3) and (g4). Then given $h \in \phi^\perp$, there exists a $\bar{t} \in \mathbb{R}$ (depending on h), such that problem (P_t) has at least two $C^{2,\alpha}$ -solutions if $t < \bar{t}$, and no solution if $t > \bar{t}$. Moreover, there is at least one solution for $t = \bar{t}$ provided we assume the following additional condition on g , in the case that $N > 3$ and the σ in (g3) is $> (N+1)/(N-1)$:

(g5) there is a function $a : \bar{\Omega} \rightarrow \mathbb{R}$, with $a(x) > 0$ if $x \in \bar{\Omega}$, such that

$$\lim_{s \rightarrow +\infty} \frac{g(x, s)}{s^\alpha} = a(x)$$

for some $(N+1)/(N-1) < \alpha < (N+2)/(N-2)$.

Remarks. 1) Condition (g2) characterizes (P_t) as a boundary value problem of the Ambrosetti Prodi type. 2) Condition (g3), together with a convenient modification of the nonlinearity $g(x, s)$ for $s < 0$, allows us to treat (P_t) by variational techniques. In this way we obtain its solutions as critical points of an associated Euler-Lagrange functional. 3) Condition (g4), in conjunction with (g2) and (g3) are used to prove that such a functional satisfies the Palais-Smale condition. 4) Condition (g5) implies an a priori bound on the solutions of (P_t) for h fixed and t in a compact set. Observe that in the cases $N = 2$ and $N > 3$ with $1 < \sigma < (N+1)/(N-1)$ an a priori bound on solutions follows from the other hypotheses of the theorem. We remark that (g5) is a strong restriction; once it is assumed, then (g3) and (g4) are automatically fulfilled for the

corresponding ranges of N and σ . The a priori bounds mentioned above are consequences of similar results for positive solutions of certain second order elliptic equations. Indeed, just observe that (P_ϵ) has a minimal solution u_ϵ and that any other solution u may be written as $u = u_\epsilon + v$, where v is a positive solution of an appropriate second order elliptic equation. The bound on the v 's follows from the work of Gidas-Spruck [7] in the case when (g5) is assumed. In the other cases the estimates of Brézis-Turner [8] will suffice.

The paper is organized as follows. In the first part we recall known results and prepare our differential problem (P_ϵ) to be treated by variational methods. In the second part we state and prove some abstract results, which will be used in the next section. Finally in the third part, we show how the abstract theorems of the previous section are used to prove results on the existence of two solutions for some second order elliptic problems. There the basic assumption is that these problems have a strict subsolution w and a strict supersolution W , with $w < W$. In particular these results imply Theorem 1. The solutions so obtained are in H_0^1 . However in view of the regularity assumptions on g and Ω made above, their regularity follows either by standard bootstrap arguments or more sophisticated ones due to Brézis-Kato [9], see also [6].

The contents of this paper were presented by the first author at the AMS Summer Institute on Nonlinear Functional Analysis and Applications in Berkeley (July 1983).

1. VARIATIONAL FORMULATION.

We start with the following result which is essentially due to Kazdan-Warner [10].

Lemma 2. Assume (g1) and (g2), and let $h \in \phi^1$ be given. Then (i) there exists a $t_1 \in \mathbb{R}$ such that (P_t) has no solution if $t > t_1$. (ii) For each given $t \in \mathbb{R}$ there exists a $C^{2,\alpha}$ -function ω_t , which is a strict subsolution of (P_t) and bounds from below all eventual supersolutions of (P_t) . (iii) There exists a $t_2 \in \mathbb{R}$ such that (P_{t_2}) has a strict supersolution $w_{t_2} \in C^{2,\alpha}(\bar{\Omega})$.

From now on we assume $h \in \phi^1$ be given and fixed. The set J of the t 's such that (P_t) has a supersolution is a halfline from $-\infty$. Let \bar{t} be the supremum of J . We assert that if $t < \bar{t}$ then (P_t) has a strict supersolution. Indeed, a supersolution of problem (P_t) , with $t < \bar{t} < \tilde{t}$, is a strict supersolution of problem $(P_{\tilde{t}})$.

Hereafter let $t < \bar{t}$ be fixed. Our goal is to prove that (P_t) has two solutions. We remark at this point that the method of monotone iteration yields readily the existence of one solution. This was the approach taken by Kazdan-Warner [10], and also used by Amann-Hess [11], Dancer [4], Berestycki [12] and one of the authors [6] to obtain the first solution (the minimal one) of the Ambrosetti-Prodi problem. The second solution was obtained by degree arguments, in the case of [4], [11] and [12], and by a variational argument, in the case of [6]. In the present paper we proceed solely by variational techniques, even for the first solution. In this way we are in the spirit of the work of Hofer [2].

In order to formulate variationally our problem, we recall [see Lemma 2] that all eventual solutions of (P_t) are bounded from below by ω_t . We remark that ω_t is the solution of the Dirichlet problem

$$-\Delta \omega_t = \mu \omega_t - C + t\phi + h \text{ in } \Omega, \quad \omega_t = 0 \text{ on } \partial\Omega,$$

where $\mu < \lambda_1$ and $C > 0$ are chosen in such a way that $g(x,s) > \mu s - C$ for all $x \in \Omega$ and $s \in \mathbb{R}$. [Such a lower bound on g exists as a consequence of (g2), cf. [6]]. So ω_t depends only on this lower bound for $g(x,s)$ and consequently a modification of g maintaining it produces the same ω_t . Thus we modify the nonlinearity $g(x,s)$ for $s < \min \omega_t$, so as to preserve the inequality $g(x,s) > \mu s - C$. The solutions of (P_t)

with this modified nonlinearity are the same as the ones of the original problem. In addition, we do this modification in such a way that the new g is still a C^1 -function and moreover $g(x,s) > 0$ for large negative s and g has linear growth as $s \rightarrow -\infty$. With this new nonlinearity the functional Φ below is well defined in H_0^1 and satisfies the Palais-Smale condition, cf. [6]:

$$\Phi(u) = \int \frac{1}{2} |\nabla u|^2 - G(x,u) - (t\phi + h)u .$$

2. ABSTRACT RESULTS.

Let H be a real Hilbert space, with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. A functional $\Phi : H \rightarrow \mathbb{R}$ of class C^1 is said to satisfy the Palais Smale condition if every sequence (u_n) in H , such that $\Phi(u_n)$ is bounded and $\Phi'(u_n) \rightarrow 0$, is precompact. The following result was a proposition proved by Hofer in [2] using the deformation lemma adapted for closed convex subsets of a Hilbert space. We provide below a different proof.

Proposition 3. Let $\Phi \in C^1(H, \mathbb{R})$ satisfy the Palais Smale condition. Let C be a closed convex subset of H . Suppose that $K \equiv I - \Phi'$ maps C into C and that Φ is bounded below in C . Then there is $u_0 \in C$ such that $\Phi'(u_0) = 0$ and $\inf_C \Phi = \Phi(u_0)$.

Proof. By Ekeland's variational principle, given $\varepsilon > 0$ there is $u_\varepsilon \in C$ such that

$$\begin{aligned} \Phi(u_\varepsilon) &\leq \inf_C \Phi + \varepsilon \text{ and} \\ (1) \quad \Phi(u_\varepsilon) &\leq \Phi(u) + \varepsilon \|u - u_\varepsilon\|, \quad \forall u \in C. \end{aligned}$$

Take in (1) $u = (1 - t)u_\varepsilon + tKu_\varepsilon$, with $0 \leq t \leq 1$, and use Taylor's formula to expand

$$\begin{aligned} \Phi(u_\varepsilon + t(Ku_\varepsilon - u_\varepsilon)) \text{ about } u_\varepsilon. \text{ We then obtain} \\ t\|\Phi'(u_\varepsilon)\|^2 \leq \varepsilon t\|\Phi'(u_\varepsilon)\| + o(t) \end{aligned}$$

from which follows that $\|\Phi'(u_\varepsilon)\| \leq \varepsilon$, making $t \rightarrow 0$. Finally use the Palais-Smale condition to conclude that there exists $u_0 \in C$ such that $u_\varepsilon \rightarrow u_0$, for some sequence $\varepsilon \rightarrow 0$. The continuity of Φ and Φ' completes the proof. \square

An operator $T : H \rightarrow H$ is said to be of type S^+ if for every sequence $u_n \rightarrow u$ (\rightarrow means weak convergence) such that $\limsup (Tu_n, u_n - u) \leq 0$, it follows that $u_n \rightarrow u_0$.

Proposition 4. Let $\Phi \in C^1(H, \mathbb{R})$ be such that Φ' is of type S^+ . Suppose that Φ is bounded below in a ball \bar{B} . Then there exists $v_0 \in \bar{B}$ such that $\Phi(v_0) = \inf_{\bar{B}} \Phi$ and $\Phi'(v_0) = \lambda v_0$, with $\lambda \leq 0$.

Proof. By Ekeland's variational principle, given $\varepsilon > 0$ there exists $v_\varepsilon \in \bar{B}$ such that

$$\begin{aligned} \Phi(v_\varepsilon) &\leq \inf_{\bar{B}} \Phi + \varepsilon \text{ and} \\ (2) \quad \Phi(v_\varepsilon) &\leq \Phi(v) + \varepsilon \|v - v_\varepsilon\|, \quad \forall v \in \bar{B}. \end{aligned}$$

Now let $u \in \bar{B}$ be arbitrary and take in (2) $v = v_\varepsilon + t(u - v_\varepsilon)$, with $0 \leq t \leq 1$. Use Taylor's formula to expand $\Phi(v_\varepsilon + t(u - v_\varepsilon))$ about v_ε and obtain, making $t \rightarrow 0$:

$$(3) \quad 0 \leq \langle \Phi'(v_\varepsilon), u - v_\varepsilon \rangle + \varepsilon \|u - v_\varepsilon\|, \quad \forall u \in \bar{B}.$$

Next taking in (3) $u = v_0$, where v_0 is the weak limit of v_ε for some sequence $\varepsilon \rightarrow 0$, and making $\varepsilon \rightarrow 0$, one obtains $\limsup_{\varepsilon \rightarrow 0} \langle \phi'(v_\varepsilon), v_\varepsilon - v_0 \rangle \leq 0$. Since ϕ' is of type S^+ , it follows that $v_\varepsilon \rightarrow v_0$. We obtain readily that $\phi(v_0) = \inf_{\bar{B}} \phi$. On the other hand, from (3) it follows that $0 < \langle \phi'(v_0), u - v_0 \rangle$ for all $u \in \bar{B}$. This implies that there exists $\lambda < 0$ such that $\phi'(v_0) = \lambda v_0$. \square

The Mountain Pass Theorem of Ambrosetti-Rabinowitz [13] states that a functional $\phi \in C^1(H, \mathbb{R})$, which satisfies the Palais-Smale condition and such that

$$(4) \quad \inf\{\phi(u) : \|u\| = r\} > \max\{\phi(0), \phi(e)\} \equiv a,$$

where $0 < r < \|e\|$, has necessarily a critical point u_0 at the level

$$(5) \quad c \equiv \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \phi(\gamma(t))$$

where $\Gamma = \{\gamma \in C^0([0,1], H) : \gamma(0) = 0, \gamma(1) = e\}$. Clearly $c > a$, which implies that $u_0 \neq 0$ and $u_0 \neq e$. What happens however if the inequality in (4) is replaced by equality? In this direction, Pucci-Serrin [14] observed that if there are numbers r and R such that $0 < r < R < \|e\|$, and

$$(6) \quad \inf\{\phi(u) : r < \|u\| < R\} > \max\{\phi(0), \phi(e)\} \equiv a$$

then c as defined in (5) is a critical value for ϕ . Moreover if $c = a$ then there is a critical point at every sphere $\partial B_\rho = \{u \in H : \|u\| = \rho\}$ for $r < \rho < R$. See also Willem [15] for an interesting proof of this fact. Rabinowitz in [3] proved a stronger form of the dual Mountain Pass Theorem, which provides an improved form of the Pucci-Serrin result. In his result, Rabinowitz replaces condition (6) by the much weaker condition $\inf\{\phi(u) : \|u\| = r\} > \phi(0) > \phi(e)$ with $\|e\| > r$, and proves that there is a critical point at the level

$$b \equiv \sup_{U \in \mathcal{O}} \inf_{u \in \partial U} \phi(u),$$

where $\mathcal{O} = \{U \subset H : U \text{ is open, } 0 \in U, e \in \bar{U}\}$. Moreover if $b = \phi(0)$ then there is a critical point at ∂B_r . The proposition below is essentially this result, but its proof differs from the one given in [3].

Proposition 5. Let $\phi \in C^1(H, R)$ satisfy the Palais Smale condition. Suppose that

$$(7) \quad \inf\{\phi(u) : \|u\| = r\} > \max\{\phi(0), \phi(e)\},$$

where $0 < r < \|e\|$. Then ϕ has a critical point $u_0 \neq 0$.

Proof. It suffices to consider the case where we have equality in (7), since the case of inequality is the usual Mountain Pass Theorem. (i) First suppose that $\phi(0) < \phi(e)$. Then either e is a local minimum (and we are through) or there is an e' near e such that $\phi(e') < \inf\{\phi(u) : \|u\| = r\}$. In the latter case we use the usual Mountain Pass Theorem to conclude. (ii) Next suppose that $\phi(0) = \phi(e)$. As in (i) above either e is a local minimum or there is e' near e with $\phi(e') < \phi(0)$. (iii) So we may always suppose from now on that $\phi(0) > \phi(e)$. Also we may assume that

$$\phi(0) = \inf_{B_r} \phi.$$

Otherwise we would apply the usual Mountain Pass Theorem to conclude. (iv) Now

$0 < \rho < r$. We claim that either there is an $\alpha > \phi(0)$ such that

$\inf\{\phi(u) : \|u\| = \rho\} > \alpha$ or there is a critical point of ϕ on ∂B_ρ . If the first

alternative occurs we apply the usual Mountain Pass Theorem and the proof is complete. To prove the claim, let us suppose that there is no such an α . Then there are $u_n \in \partial B_\rho$ such that

$$\phi(u_n) < \phi(0) + \frac{1}{n}$$

Consider the ring $R = \{u \in H : 0 < \rho - \eta < \|u\| < \rho + \eta < r\}$ for an appropriate $\eta > 0$.

Now use Ekeland variational principle for ϕ in the ring. Thus there are $v_n \in R$ such that

$$(8) \quad \phi(v_n) < \phi(u_n), \quad \|v_n - u_n\| < \frac{1}{n},$$

$$\phi(v_n) < \phi(u) + \frac{1}{n} \|u - v_n\|, \quad \forall u \in R.$$

For large n , v_n is in the interior of R . So take in (8) $u = v_n + tu$, where $t \in R$ is small and u is arbitrary with $\|u\| = 1$. Using Taylor's formula to expand $\phi(v_n + tu)$ about v_n and letting $t \rightarrow 0$ we obtain $\|\phi'(v_n)\| < \frac{1}{n}$. Finally using the Palais-Smale condition we conclude that $v_n + u_0 \in \partial B_\rho$ and u_0 is a critical point of ϕ . \square

3. APPLICATION TO ELLIPTIC PROBLEMS.

Since the abstract results presented in the previous section allow us to treat a larger class of second order elliptic problems we start anew the abstract setting of the differential problem. We observe that for the use of variational methods and the existence of H_0^1 -solutions a great deal less of regularity is required from both the nonlinearity g and the linear part of the equation. We also remark that the basic assumption is the existence of an ordered pair of a subsolution and a supersolution. So besides the Ambrosetti-Prodi problem stated in the Introduction we can treat other problems where such a pair is produced.

Consider the second order elliptic operator

$$Lu = - \sum_{i,j=1}^N D_j(a_{ij}(x)D_i u) + c(x)u$$

where $a_{ij} \in L^\infty(\Omega)$, $c \in L^{N/2}(\Omega)$ and $c(x) > 0$ a.e. Ellipticity here means that there is a constant $c > 0$ such that $\sum a_{ij}(x)\xi_i\xi_j > c|\xi|^2$. Let $a : H_0^1 \times H_0^1 \rightarrow \mathbb{R}$ be the bilinear form associated to L and defined by

$$a[u,v] = \int \sum a_{ij}(x)D_i u D_j v + c(x)uv.$$

We assume that $g : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function such that

$$|g(x,s)| \leq c|s|^\sigma + d(x)$$

where $\sigma = (N+2)/(N-2)$ and $d \in L^{2N/(N+2)}$ if $N \geq 3$, or $d \in L^p$ and $1 < p, \sigma < \infty$ if $N = 2$. We then consider the Dirichlet problem

$$(9) \quad \begin{cases} a[u,v] = \int g(x,u)v, & \forall v \in H_0^1 \\ u \in H_0^1 \end{cases}$$

Let $\Phi : H_0^1 \rightarrow \mathbb{R}$ be the corresponding Euler-Lagrange functional defined by

$$(10) \quad \Phi(u) = \frac{1}{2} a[u,u] - \int G(x,u)$$

where $G(x,s) = \int_0^s g(x,\xi)d\xi$. We renorm H_0^1 with the norm $\|u\| = a[u,u]^{1/2}$; denote the

corresponding inner product by $\langle \cdot, \cdot \rangle : \langle u, v \rangle = a[u, v]$. Clearly ϕ is C^1 and its derivative at u is given by

$$\langle \phi'(u), v \rangle = \langle u, v \rangle - \int g(x, u) v, \quad \forall v \in H_0^1.$$

We then see that the critical points of ϕ are precisely the solutions of (9). Let us define the map $K : H_0^1 \rightarrow H_0^1$ by

$$\langle Ku, v \rangle = \int g(x, u) v, \quad \forall v \in H_0^1.$$

Theorem 6. Assume conditions on g that guarantee that ϕ defined in (10) satisfies the Palais-Smale condition. Suppose that there exist a subsolution $\omega \in H_0^1$ and a supersolution $W \in H_0^1$ of (9) such that $\omega < W$, that is,

$$(11) \quad a[\omega, v] < \int g(x, \omega) v \text{ and } a[W, v] > \int g(x, W) v, \quad \forall v \in H_0^1, v > 0.$$

Assume also that $g(x, s)$ is a nondecreasing function of s for $\omega(x) < s < W(x)$. Then there exists $u_0 \in H_0^1$ such that $u_0 \in [\omega, W]$, $\phi(u_0) = \inf_{[\omega, W]} \phi$ and $\phi'(u_0) = 0$. [Consequently u_0 is a solution of (9)].

Remarks. 1) We use the notation $[\omega, W]$ to denote the set of $u \in H_0^1$ such that $\omega(x) < u(x) < W(x)$ for $x \in \Omega$ (a.e.).

2) The set of assumptions on g prescribed at the Introduction are sufficient to guarantee that ϕ satisfies the Palais-Smale condition.

Proof. (i) $C = [\omega, W]$ is a closed and convex subset of H_0^1 . (ii) K maps C into C .

Indeed for $u \in [\omega, W]$, we have $g(x, \omega) < g(x, u) < g(x, W)$, which implies

$a[\omega, v] < a[Ku, v] < a[W, v]$ for all $v \in H_0^1, v > 0$. By the Maximum Principle (see Gilbarg-

Trudinger [16, p. 168]) it follows that $\omega < Ku < W$. (iii) ϕ is bounded below in C .

Indeed, if $u \in [\omega, W]$ then $|u(x)| < |w(x)| + |W(x)|$, which implies that

$\|u\|_{L^{2N/(N-2)}} < C$. So $\int |G(x, u)| < C$, which proves the claim. (iv) Finally the

conclusion of the theorem follows readily from Proposition 3. \square

Corollary 7. Same hypotheses of Theorem 6, except the assumption that g is nondecreasing in s for $w(x) < s < W(x)$. Assume however that g is locally lipschitzian and w and W are continuous in $\bar{\Omega}$. Then the same conclusions of Theorem 6 hold.

Proof. In view of the hypotheses, there is a positive constant M such that the

function $g_M(x, s) \equiv g(x, s) + Ms$ is nondecreasing in s for $w(x) < s < W(x)$. The

Dirichlet problem

$$(12) \quad \begin{cases} a_M[u,v] = \int g_M(x,u)v, & \forall v \in H_0^1 \\ u \in H_0^1 \end{cases}$$

where $a_M[u,v] = a[u,v] + M \int uv$, has the same solutions as problem (9). Moreover the functional associated to (12) is the same as Φ . And we may apply Theorem 6 to problem (12). \square

Theorem 8. Assume conditions on g that guarantee that Φ satisfies the Palais-Smale condition and Φ' is of type S^+ . Suppose that g is locally lipschitzian and the coefficients of L are such that the results of the L^p -regularity theory hold true. Assume also that $w \in C^{1,\alpha}$ is a strict subsolution and $W \in C^{1,\alpha}$ is a strict supersolution, with $w < W$. Then there exists $u_0 \in [w,W]$ such that $\Phi'(u_0) = 0$ and u_0 is a local minimum of Φ . That is, there exists $\varepsilon > 0$, such that

$$\Phi(u_0) = \inf_{B_\varepsilon(u_0)} \Phi$$

Remark. The Ambrosetti-Prodi problem stated in the Introduction satisfies all the hypotheses of the above theorem.

Proof. From Theorem 6 it follows that there is $u_0 \in [w,W]$ such that $\Phi'(u_0) = 0$ and

$$(13) \quad \Phi(u_0) = \inf_{[w,W]} \Phi.$$

It can be shown that $u_0 \in C^{1,\alpha}$; the first step in the proof of this fact uses arguments from Brézis-Kato [9] to show that $u_0 \in L^p$ for $p > 2N/(N-2)$, see [6]. Then a bootstrap argument is used. Suppose by contradiction that u_0 is not a local minimum. Then for every $\varepsilon > 0$ there is $u_\varepsilon \in \overline{B_\varepsilon(u_0)}$ such that $\Phi(u_\varepsilon) < \Phi(u_0)$. By Proposition 4 there is $v_\varepsilon \in \overline{B_\varepsilon(u_0)}$ such that

$$(14) \quad \Phi(v_\varepsilon) = \inf_{B_\varepsilon(u_0)} \Phi < \Phi(u_\varepsilon) < \Phi(u_0), \text{ and}$$

$$(15) \quad \Phi'(v_\varepsilon) = \lambda_\varepsilon (v_\varepsilon - u_0), \text{ with } \lambda_\varepsilon < 0.$$

By the same arguments used to establish that $u_0 \in C^{1,\alpha}$, we prove that $v_\epsilon \in C^{1,\alpha}$. The strong maximum principle and the fact that w and W are not solutions imply that $v_\epsilon \in [w, W]$ for small ϵ . Then (14) contradicts (13). \square

Corollary 9. In addition to all the hypotheses of Theorem 8, assume that there is $v \in H_0^1$ such that $|v| > \max\{|w|, |W|\}$ and $\phi(v) < \min\{\phi(w), \phi(W)\}$. Then the Dirichlet problem (9) has two solutions.

Remark. The existence of such a v is trivial in the case that ϕ is not bounded below in H_0^1 . This is always the case for superlinear problems.

Proof. From Theorem 8 it follows the existence of the first solution, which is a local minimum of ϕ . The second solution is a consequence of Proposition 5. \square

REFERENCES

- [1] Ekeland, I. - "Non convex minimization problems". Bull. AMS 1 (1979), pp. 443-474.
- [2] Hofer, H. - "Variational and Topological Methods in Partially Ordered Hilbert Spaces". Math. Ann. 261 (1982), pp. 493-514.
- [3] Rabinowitz, P. - "Some aspects of critical point theory". MRC Tech. Rep. #2465 (Jan. 1983).
- [4] Dancer, E. N. - "On the ranges of certain weakly nonlinear elliptic partial differential equations". J. Math. Pures et Appl. 57 (1978), pp. 351-366.
- [5] Chang, K. C. - personal communication.
- [6] de Figueiredo, D. G. - "On the superlinear Ambrosetti-Prodi problem". MRC Tech. Rep. #2522 (May 1983).
- [7] Gidas, B. and Spruck, J. - "A priori bounds for positive solutions of nonlinear elliptic equations". Comm. PDE 6 (1981), pp. 883-901.
- [8] Brézis, H. and Turner, R.E.L. - "On a class of superlinear elliptic problems". Comm. PDE 2 (1977), pp. 601-614.
- [9] Brézis, H. and Kato, T. - "Remarks on the Schrödinger operator with singular complex potentials". J. Math. Pures et Appl. 58 (1979), pp. 137-151.
- [10] Kazdan, J. and Warner, F. W. - "Remarks on some quasilinear elliptic equations". Comm. Pure Appl. Math. XXVIII (1975), pp. 567-597.
- [11] Amann, H. and Hess, P. - "A multiplicity result for a class of elliptic boundary value problems". Proc. Royal Soc. Edinburgh 84A (1979), pp. 145-151.
- [12] Berestycki, H. - "Le nombre de solutions de certains problèmes semilinéaires elliptiques". J. Fct. Anal.
- [13] Ambrosetti, A. and Rabinowitz, P. - "Dual variational methods in critical point theory and applications". J. Fctl. Anal. 14 (1973), pp. 349-381.
- [14] Pucci, P. and Serrin, J. - "A mountain pass theorem". To appear.
- [15] Willem, M. - "Lectures on Critical Point Theory". Trabalho de Matemática n°199. Universidade de Brasília (Feb. 1983).
- [16] Gilbarg, D. and Trudinger, N. S. - "Elliptic Partial Differential Equations". Springer Verlag (1977).

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